An Interval Finite Difference Method for Solving the One-Dimensional Heat Equation

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Abstract. In the paper the interval methods for solving a parabolic partial differential equation known as the heat equation is considered. The authors use the finite difference schemes to obtain the required interval formulas and use the interval realization of very effective variety of direct Cholescy method for solving the system of interval linear equations of the special positive definite, tridiagonal and symmetric matrices. The interval methods shown in the paper include the error term of the corresponding conventional methods and together with interval floating-point arithmetic allow the user to obtain a guaranteed result. The theoretical considerations are also supported by results of some numerical experiments.

Key words: interval methods, finite difference methods, partial differential equations, heat equation

1 Introduction

Since the 1960s when first interval methods were proposed by Ramon E. Moore (see [8]) verified computing became well-known and eagerly developed. Nowadays many different one- and multi-dimensional problems (e.g. nonlinear root-finding, solving of linear and nonlinear systems of equation, linear and global optimization) can be solved by interval methods (see [4], [6]). There are also many different algorithms for finding the solution of the initial value problem for ordinary differential equations. Let us note that the boundary value problems for the partial differential equations belong to a large class of great importance problems in many scientific fields Since there are few papers devoted to the interval methods for solving such problems, then the authors find it interesting to take up the research in the topic considered.

2 Heat Equation and Finite Difference Methods

The parabolic partial differential equation we consider is the one-dimensional heat equation of the form

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < L,\ t > 0, \tag{1}$$

subject to the boundary conditions

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$
$$u(0,t) = u(L,t) = 0, \quad t > 0. \tag{3}$$

Before we adapt the finite difference method for the boundary value problem (1)-(3), we set the maximum time T, choose two integers n and m and find the mesh constants h and k such as $h = L/n$ and $k = T/m$. Hence the grid points are (x_i, t_j), where $x_i = ih$ for $i = 0, 1, \ldots, n$ and $t_j = jk$ for $j = 0, 1, \ldots, m$. Then using the Taylor series for each interior grid point (see e.g. [1], [3]) we obtain the backward difference formula for $(\partial u/\partial t)(x_i, t_j)$ in the form

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i,t_j) - u(x_i,t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j), \tag{4}$$

and the centered difference formula for $\left(\partial^2 u/\partial x^2\right)(x_i, t_j)$ given by

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_{i-1},t_j) - 2u(x_i,t_j) + u(x_{i+1},t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j), \tag{5}$$

where $\eta_j \in (t_{j-1}, t_j)$ and $\xi_i \in (x_{i-1}, x_{i+1})$. Since we can express (1) at the grid points (x_i, t_j), $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$, then substituting (4) and (5) into (1) gives

$$\begin{aligned} &\frac{u(x_i,t_j) - u(x_i,t_{j-1})}{k} - \alpha^2 \frac{u(x_{i-1},t_j) - 2u(x_i,t_j) + u(x_{i+1},t_j)}{h^2} \\ &\qquad = -\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_j) - \alpha^2 \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j), \\ &i = 1, 2, \ldots, n-1,\ j = 1, 2, \ldots, m, \end{aligned} \tag{6}$$

with the boundary conditions

$$\begin{aligned} u(x_i,0) &= f(x_i), \quad i = 0, 1, \ldots, n, \\ u(0,t_j) &= u(L,t_j) = 0, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{7}$$

Now let us denote $\lambda = \alpha^2\left(k/h^2\right)$ and let $u_{i,j}$ approximate $u(x_i, t_j)$. If we neglect the error term then from (6) we get the backward difference method with local truncation error $O\left(k + h^2\right)$ of the form

$$(1+2\lambda)\,u_{i,j} - \lambda u_{i-1,j} - \lambda u_{i+1,j} = u_{i,j-1}, \quad i = 1, 2, \ldots, n-1,\ j = 1, 2, \ldots, m, \tag{8}$$

where

$$u_{i,0} = f(x_i),\ i = 0, 1, \ldots, n, \quad u_{0,j} = u_{n,j} = 0,\ j = 1, 2, \ldots, m. \tag{9}$$

3 An Interval Finite Difference Method

Let us consider the finite difference formulas (4) and (5). The problem is how to find intervals that contain $\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j)$ and $\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)$, where $\eta_j \in (t_{j-1}, t_j)$ and $\xi_i \in (x_{i-1}, x_{i+1})$. From (1) we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \frac{\partial^3 u}{\partial t \partial x^2}(x,t), \quad \frac{\partial^4 u}{\partial x^4}(x,t) = \frac{1}{\alpha^2}\frac{\partial^3 u}{\partial t \partial x^2}(x,t). \tag{10}$$

Let us assume that

$$\left|\frac{\partial^3 u}{\partial t \partial x^2}(x,t)\right| \leq M, \quad 0 \leq x \leq L,\ 0 \leq t \leq T. \tag{11}$$

Then for $i = 1, 2, \ldots, n-1,\ j = 1, 2, \ldots, m$ we have

$$\frac{\partial^2 u}{\partial t^2}(x_i, \eta_j) \in \alpha^2 [-M, M], \quad \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j) \in \frac{1}{\alpha^2}[-M, M]. \tag{12}$$

Finally, substituting (12) to (6) gives an interval difference method of the form

$$\begin{aligned}(1+2\lambda) U_{i,j} - \lambda U_{i-1,j} - \lambda U_{i+1,j} = U_{i,j-1} \\ -\alpha^2 \frac{k^2}{2}[-M, M] - \frac{kh^2}{12}[-M, M], \\ i = 1, 2, \ldots, n-1,\ j = 1, 2, \ldots, m,\end{aligned} \tag{13}$$

where

$$U_{i,0} = F([ih, ih]),\ i = 0, 1, \ldots, n, \quad U_{0,j} = U_{n,j} = [0, 0],\ j = 1, 2, \ldots, m. \tag{14}$$

The interval method (13) with (14) has also the matrix representation as follows:

$$AU^{(j)} = U^{(j-1)}, \quad j = 1, 2, \ldots, m, \tag{15}$$

where

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \vdots & 0 & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & \vdots & 0 & 0 & 0 \\ 0 & -\lambda & 1+2\lambda & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 1+2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & \vdots & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & 0 & \vdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}, \tag{16}$$

$$U^{(j)} = \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ U_{3,j} \\ \cdots \\ U_{n-3,j} \\ U_{n-2,j} \\ U_{n-1,j} \end{bmatrix}, \quad U^{(j-1)} = \begin{bmatrix} U_{1,j-1} + R \\ U_{2,j-1} + R \\ U_{3,j-1} + R \\ \cdots \\ U_{n-3,j-1} + R \\ U_{n-2,j-1} + R \\ U_{n-1,j-1} + R \end{bmatrix}, \tag{17}$$

and

$$R = -\alpha^2 \frac{k^2}{2} [-M, M] - \frac{kh^2}{12} [-M, M]. \tag{18}$$

Note that the matrix A is tridiagonal and symmetric. Since $\lambda > 0$, then A is also positive defined and strictly diagonally dominant.

It can also be shown that if $u(x_i, 0) \in U_{i,0}$, $i = 0, 1, \ldots, n$ and the assumptions given above are met then $u(x_i, t_j) \in U_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m$.

4 Conclusions

The authors used the interval finite difference method considered for solving many boundary value problems of the form (1)-(3). The main conclusion is that in all examples the values of the exact solution at interior grid points belong to the appropriate interval solutions obtained. The widths of the interval solutions are compared for selected step sizes h and k. The optimal values of these mesh constants for each particular problem are chosen. The interval finite difference method proposed in the paper is the first attempt of the authors to develop effective algorithm for solving the one-dimensional heat equation. Since the heat equation is of fundamental importance in many scientific fields and the results obtained seem encouraging then the research on the problem will be continued.

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